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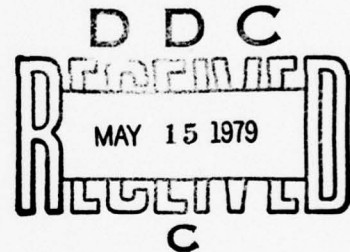
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PARTITIONED TRANSIENT ANALYSIS
PROCEDURES FOR
COUPLED-FIELD PROBLEMS

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20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

A general partitioned transient analysis procedure is proposed, which is amenable to a unified stability analysis technique. The procedure embodies two existing implicit-explicit procedures and one existing implicit-implicit procedure. A new implicit-explicit procedure is discovered, as a special case of the general procedure, that allows degree-by-degree implicit or explicit selections of the solution vector and can be implemented within the framework of the implicit integration packages. A new element-by-element implicit-implicit procedure is also presented which satisfies program modularity requirements and enables the use of single-field implicit integration packages to solve coupled-field problems.

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i

TABLE OF CONTENTS

INTRODUCTION

The transient response analysis of many engineering systems involves the
formulation of semi-discrete coupled-field equations of motion which are
then solved by direct time integration methods. Such coupled-field equa-
tions, for example, arise in the modeling of structure-medium interaction
problems, thermal-fluid flow problems, and structure-structure interaction
problems. The direct time integration of these equations presents compu-
tational difficulties that are not normally encountered in single-field
problems or in structure-structure interaction problems where mesh sizes
are more or less uniform. There are two major sources of difficulties,
one related to computational aspects and the other to implementation.

First, the isolated fields may have vastly different response character-
istics. This suggests that for computational efficiency different inte-
gration algorithms and/or different time step sizes for each solution
vector should be adopted. In fluid-structure interaction problems, for
instance, the characteristic response time scale of the fluid-field solu-
tion vector is typically much slower than that of the structure-field
vector.

Second, the bulk of existing engineering analysis software has been devel-
oped primarily for the treatment of single-field problems. One is thus
motivated to make use of existing single-field analysis software to solve
the coupled-field equations of motions; this imposes modularity constraints
on software architecture. If modularity requirements are met, the direct
time integration of the entire equations can be performed either by se-
quential or parallel execution of single-field analyzers. Solution pro-
cedures geared to this operation mode will be termed <u>partitioned transient
analysis procedures</u> (or simply <u>partitioned procedures</u>). As shown in the

1

paper, there is a multitude of possible partitioned transient analysis procedures for each coupled-field system. Therefore, the task of selecting a "best" procedure can become formidable. This is because the analysis of the stability and accuracy of partitioned procedures is much harder than that of single-field solution procedures. Furthermore, the assessment of the relative merit of competing procedures may be difficult to settle if different analysis techniques are employed for each procedure.

Partitioned procedures for structural dynamic problems were first proposed by Belytschko and Mullen [1-2]. They described an implicit-explicit (I-E) partition through which meshes that exhibit high (low) frequency response characteristics are treated by implicit (explicit) integration formulas. Hughes and Liu [3-4] proposed an alternative I-E partition, which apparently leads to a simpler implementation in finite-element programs.

Few systematic studies of partitioned procedures for specific coupled-field problems have appeared in the literature. An analysis of integration procedures for the coupled sound and heat flow problems by Morimoto [5] appears to be the first study on the subject. Recently, Park, Felippa and DeRuntz [6] studied implicit-implicit partitioned procedures for a class of fluid-structure interaction equations. Their objective was to treat the symmetric parts of both the fluid and structural equations by implicit integration formulas and coupling terms by extrapolation formulas. The two primary solution vectors were the scattered fluid pressure on the "wet" structural surface and the structural displacements. Hence, the solution of these coupled-field equations was obtained by a sequential execution of fluid and structural analyzers, which led to the term "staggered solution procedures". Unconditional stabilization of such implicit-implicit procedures was made possible by exploiting the availability of the fluid radiation damping term in the equations of motion, which is the result of treating the infinite fluid domain by a doubly asymptotic approximation [7]. The idea of introducing damping for stabilization was subsequently pursued by Belytschko, Yen and Mullen [8] in their study of an implicit-implicit procedure for a two-degree-of-freedom model problem which describes certain fluid-structure interaction problems.

2

Common to the aforementioned studies is the use of a specific partitioned procedure and/or restriction to special classes of problems. The present study endeavors to consider a general partitioned procedure which is amenable to a unified stability analysis technique. Motivation for undertaking this study is twofold: it eliminates case-by-case stability analysis of each particular partitioned procedure, and it provides a common ground for evaluating competing procedures.

As a means to achieve the preceding objectives, the present paper utilizes the following three basic formualtion steps. First, the entire semi-discrete equations of motion are treated by implicit integration formulas. Second, the resulting difference equations are partitioned. Finally, coupling terms that are transferred to the right-hand side in the difference equations are extrapolated. In other words, the first step completes the temporal discretization. The second defines computer implementation as regards matrix configurations that have to be treated in each single-field analyzer. The last step determines which parts or portion of the entire solution vector sbould be treated explicitly. It will be shown that the present formulation can embody existing partitioned procedures plus hitherto untried ones that may offer algorithmic and/or implementation advantages.

3

LINEAR COUPLED-FIELD EQUATIONS OF MOTION

We consider problems governed by the general matrix equation

$$\underset{\sim}{A}\, \underline{\ddot{u}} + \underset{\sim}{B}\, \underline{\dot{u}} + \underset{\sim}{C}\, \underline{u} = \underline{f} \tag{1a}$$

where $\underline{u}$ is the complete solution state N-vector

$$\underline{u}^T = (\underline{u}^x, \underline{u}^y, \ldots)^T \tag{1b}$$

and $\underline{u}^x$, $\underline{u}^y$... represent single-field solution state vectors; a superposed dot denotes time (t) differentiation and the superscript T on a matrix or vector denotes transposition. Matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ represent generalized mass, damping and stiffness matrices, respectively, and $\underline{f}$ is the generalized force vector.

The term "generalized" is used to emphasize that no specific form of matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$ is required for carrying out the basic three formulation steps outlined in the Introduction. For example, if $\underset{\sim}{A}$ is diagonal, $\underset{\sim}{B}$ is null, and $\underset{\sim}{C}$ is positive definite, equation (1) represents a lumped-mass and stiffness system that describes undamped oscillatory problems. If the entries in $\underset{\sim}{A}$ corresponding to $\underline{u}^y$ and those in $\underset{\sim}{B}$ corresponding to $\underline{u}^x$ are zero, equation (1) represents a coupled hyperbolic ($\underline{u}^x$) and parabolic ($\underline{u}^y$) system that describes fluid-structure interaction problems via a doubly asymptotic approximation [7]. Other coupled-field problems can be defined by appropriate selection of entries in matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, and $\underset{\sim}{C}$. Thus, the application of the basic three formulation steps to equation (1) can cover fairly general partitioned transient analysis procedures applicable to a wide class of coupled-field problems.

IMPLICIT INTEGRATION,
GENERAL PARTITIONING AND EXTRAPOLATION

Implicit linear multistep formulas can be expressed in a compact form (see Felippa and Park [9] for details):

$$\underline{u}_n = \delta \, \underline{\dot{u}}_n + \underline{h}_n^u \tag{2}$$

where $\underline{h}_n^u$ represents the contribution of historical terms and δ is a formula-dependent generalized step size. It is noted that each field solution state vector in equation (1b) can be integrated by different integration formulas. Therefore, equation (2) is to be viewed as a symbolic representation of these different integration formulas, viz., for $\underline{u}^T = (\underline{u}^x, \underline{u}^y, \ldots)^T$, we have $\delta = (\delta_x, \delta_y, \ldots)$, and similarly for the historical vector $\underline{h}_n^u$. Substitution of equation (2) into equation (1) yields

$$\underset{\sim}{E} \, \underline{u}_n = \underline{g}_n \tag{3}$$

where
$$\underset{\sim}{E} = \underset{\sim}{A} + \delta \, \underset{\sim}{B} + \delta^2 \, \underset{\sim}{C} \tag{4}$$

$$\underline{g}_n = \delta^2 \, \underline{f}_n + (\underset{\sim}{A} + \delta \, \underset{\sim}{B}) \, \underline{h}_n^u + \delta \, \underset{\sim}{A} \, \underline{h}_n^{\dot{u}} \tag{5}$$

Now we wish to solve the difference equation (3) for the entire coupled-field equations by making use of single-field analysis software. To this end we restrict our considerations to two-field problems and consider the following general matrix partitioning:

$$\underset{\sim}{E} = \underset{\sim}{E}_x + \underset{\sim}{E}_y \tag{6}$$

where
$$\underset{\sim}{E}_x = \underset{\sim}{A} + \delta \, \underset{\sim}{B}_x + \delta^2 \, \underset{\sim}{C}_x \tag{7}$$

$$\underset{\sim}{E}_y = \delta \, \underset{\sim}{B}_y + \delta^2 \, \underset{\sim}{C}_y \tag{8}$$

Equation (3) is rearranged using Equation (6) in the form

$$\underset{\sim}{E}_x \, \underline{u}_n \; = \; g_n \, - \, \underset{\sim}{E}_y \, \underline{u}_n^{(P)} \tag{9}$$

where $\underline{u}_n^{(P)}$ is a suitable extrapolator for $\underline{u}_n$.

We call the formulation as given by equations (2) through (9) an <u>implicit-explicit</u> (I-E) procedure if $\underset{\sim}{E}_y$ contains non-zero diagonal entries, and an <u>implicit-implicit</u> (I-I) procedure if $\underset{\sim}{E}_y$ contains only off-diagonal entries.

We use here the term <u>partitioning</u> in preference to <u>splitting</u> for the following reason. The term "matrix splitting" is often associated with procedures in which one wishes to reduce the bandwidth of the semi-implicit iteration matrix or to treat directional-sensitive matrix coefficients separately in alternating direction methods [10-12]. Thus, in splitting procedures one solves for the entire solution state vector simultaneously. On the other hand, in partitioned procedures one solves first for part of the entire solution state vector, which is then used to solve a portion of the remaining solution vector. This distinction can be clarified by the following examples which arise in structural dynamics.

EXAMPLES OF PARTITIONING

For reasons discussed later, matrix $\underset{\sim}{A}$ is __not__ partitioned and the same partitioning is applied to $\underset{\sim}{B}$ and $\underset{\sim}{C}$. Hence, we will deal only with matrix $\underset{\sim}{C}$ in the following examples.

Let us represent the matrix $\underset{\sim}{C}$ as a 3x3 block matrix:

$$\underset{\sim}{C} = \begin{bmatrix} \underset{\sim}{C}_{xx} & \underset{\sim}{C}_{xb} & \underset{\sim}{0} \\ \underset{\sim}{C}_{bx} & \underset{\sim}{C}_{bb} & \underset{\sim}{C}_{by} \\ \underset{\sim}{0} & \underset{\sim}{C}_{yb} & \underset{\sim}{C}_{yy} \end{bmatrix} \tag{10}$$

and the corresponding solution state vector $\underset{\sim}{u}$ by

$$u^T = (\underset{\sim}{u}^x, \underset{\sim}{u}^b, \underset{\sim}{u}^y)^T \tag{11}$$

where super or subscripts x and y designate the two domains, and b designates the boundary as shown in Figure 1. We now illustrate some useful partitions.

__Example 1__: Belytschko and Mullen Node-by-Node Partitioning [1-2]. This partitioning is achieved from equation (10) by grouping the boundary contributions to either one of the domains and choosing $\underset{\sim}{C}_y$ as:

$$\underset{\sim}{C}_y = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{C}_{yb} & \underset{\sim}{C}_{yy} \end{bmatrix} \tag{12}$$
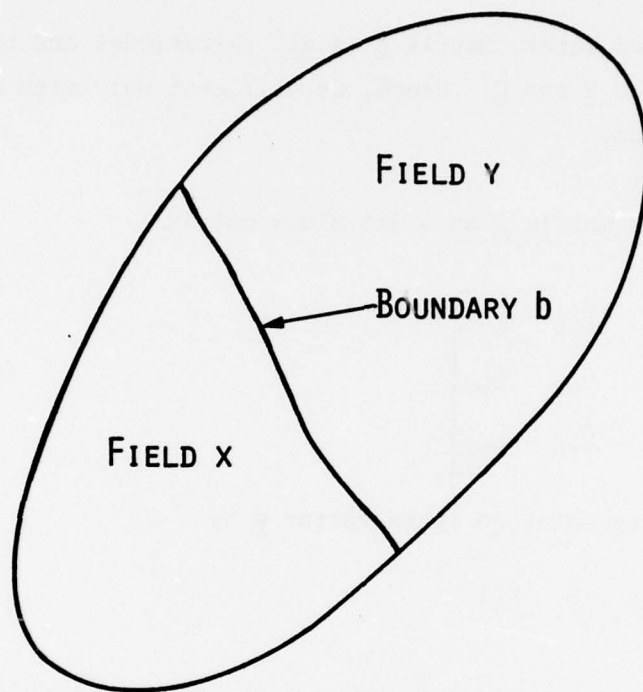
7

Figure 1. Coupling of Two Fields with Common Boundaries.

This partition was used in an implicit-explicit procedure. Partition (12) enables us to obtain the entire solution vector $\underline{u}$ in two separate solution stages. First, the solution vector $\underline{u}^y$ is obtained explicitly as $\underline{A}$ is assumed diagonal. Then the remaining vector $(\underline{u}^x, \underline{u}^b)^T$ is obtained by solving the implicit algebraic equations once $\underline{u}^y$ is available.

Example 2:   Hughes and Liu Element-by-Element Partitioning [3-4].

If we decompose the boundary matrix $\underset{\sim}{C}_{bb}$ in equation (10) into two element-level matrices according to the two groups of boundary-adjacent finite elements, the following partitioning for $\underset{\sim}{C}_y$ results:

$$
\underset{\sim}{C}_y = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\[6pt] \underset{\sim}{0} & \underset{\sim}{C}_{bb}^y & \underset{\sim}{C}_{by} \\[6pt] \underset{\sim}{0} & \underset{\sim}{C}_{yb} & \underset{\sim}{C}_{yy} \end{bmatrix}
\tag{13}
$$

where $\underset{\sim}{C}_{bb} = \underset{\sim}{C}_{bb}^x + \underset{\sim}{C}_{bb}^y$, in which $\underset{\sim}{C}_{bb}^x$ is the contribution of the element-level matrices that belong to elements in the spatial domain x, and similarly for $\underset{\sim}{C}_{bb}^y$.

Example 3: A New DOF-by-DOF Partitioning.

A partitioning suitable for working at the degree-of-freedom (DOF) level results if we take

$$
\underset{\sim}{C}_y = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\[6pt] \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\[6pt] \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{C}_{yy} \end{bmatrix}
\tag{14}
$$

It will be shown later on in the paper that this partition gives the same stability condition as the node-by-node partition (12). Nevertheless, this DOF-by-DOF partition (14) offers one computational advantage over the node-by-node partition: it can be implemented within a single-field

9

implicit integration package as the implicit-portion of the stiffness matrix remains symmetric. On the other hand, the node-by-node partition (12) requires that integration be carried out in two separate integration packages: explicit and implicit, because the implicit-portion stiffness matrix is not symmetric.

STABILITY OF GENERAL PARTITIONED PROCEDURE

The stability of the general partitioned equation (9) can be examined by seeking nontrivial solutions for $\underline{f}_n = \underline{0}$ in the form:

$$\underline{u}_n = \lambda \, \underline{u}_{n-1} \tag{15}$$

where λ is the solution amplification factor. The integration formula (2) can now be expressed in terms of its associated characteristic polynomial (see Appendix A):

$$[\rho(\lambda) - \delta \, \sigma(\lambda)] \, \underline{u}_{n-m} = \underline{0} \tag{16}$$

and similarly for the extrapolator (A.2):

$$\underline{u}_n^{(p)} = e(\lambda) \, \underline{u}_{n-m} \tag{17}$$

where the subscript m denotes the number of step intervals used in the formula (2). Substituting Equations (16) and (17) into Equation (9) one obtains the characteristic system:

$$\underset{\sim}{J}(\lambda) \, \underline{u}_{n-m} = \underline{0} \tag{18}$$

where
$$\underset{\sim}{J}(\lambda) = [\rho^2 \, \underset{\sim}{A} + \delta \, \rho \, \sigma \, \underset{\sim}{B}_x + \delta \, \rho \, (\sigma - \lambda^m + e) \, \underset{\sim}{B}_y$$
$$+ \, \delta^2 \, \sigma^2 \, \underset{\sim}{C}_x + \delta^2 (\sigma^2 - \lambda^m \rho + \rho \, e) \, \underset{\sim}{C}_y] \tag{19}$$

in which ρ stands for $\rho(\lambda)$, etc.

11

Remark: It is important that velocities be updated by the formula

$$\dot{\underline{u}}_n = \delta \, \ddot{\underline{u}}_n + \underline{h}_n^{\dot{u}} \tag{20}$$

and not by the differentiation formula

$$\dot{\underline{u}}_n = (\underline{u}_n - \underline{h}_n^u) / \delta \tag{21}$$

in order to avoid numerical instability (see Appendix A).

For algebraic convenience we introduce the transformation:

$$\lambda = \frac{1 + z}{1 - z} \tag{22}$$

which maps the solution-bounding unit disk $|\lambda| \leq 1$ onto the entire left-hand z-plane. The characteristic equation in the z variable is

$$\det \underline{J}(\frac{1 + z}{1 - z}) = 0 \tag{23}$$

For stability the roots of the characteristic equation (23) must satisfy:

$$Re(z_i) \leq 0, \; i = 1,\ldots\ldots\ldots, 2mN \tag{24}$$

where N is the size of equation (1) and m is the number of steps entering in the integration formula (2).

The major thrust of the present paper is to treat various partitioned procedures and their stability conditions by a general partitioned procedure (9) and a unified stability analysis. To this end we will limit ourselves to the use of the trapezoidal formula and one term extrapolator in the sequel, viz:

$$\rho(\lambda) = \lambda - 1$$

$$\sigma(\lambda) = \lambda + 1$$

$$\delta = 0.5\ h \qquad (25)$$

$$e(\lambda) = 1$$

so that equations (19) and (23), respectively, reduce to

$$\underset{\sim}{J}_T(\lambda) = (\lambda - 1)^2\ \underset{\sim}{A} + \delta(\lambda^2 - 1)\ \underset{\sim}{B}_x + \delta^2(\lambda + 1)^2\ \underset{\sim}{C}_x$$
$$+ 2\ \delta(\lambda - 1)\ \underset{\sim}{B}_y + 4\delta^2\lambda\underset{\sim}{C}_y \qquad (26)$$

$$\det\ \left[(\underset{\sim}{A} - \delta\ \underset{\sim}{B}_y - \delta^2\ \underset{\sim}{C}_y)\ z^2 + \delta\ \underset{\sim}{B}\ z + \delta^2\ \underset{\sim}{C}\right] = 0 \qquad (27)$$

The stability of the general partitioned procedure (9), when the trapezoidal
integration formula and one term extrapolator are used, can now be examined
via the characteristic z-polynomial equation (27) subject to the constraints
(24). As basic means for evaluating the stability condition (24) we cite
the classical theorems of Routh-Hurwitz and Lyapunov (see Chapter XV of
Gantmacher [13]). For the case in which all matrices appearing in equation
(27) are symmetric, it is convenient to use a theorem of Bellman [14].

<u>Theorem 1 (Bellman).</u> If $(\underset{\sim}{A} - \delta\ \underset{\sim}{B}_y - \delta^2\ \underset{\sim}{C}_y)$, $\underset{\sim}{B}$ and $\underset{\sim}{C}$ are non-negative defin-
ite, and either $(\underset{\sim}{A} - \delta\ \underset{\sim}{B}_y - \delta^2\ \underset{\sim}{C}_y)$ or $\underset{\sim}{C}$ is positive definite, then equation
(27) has no roots with positive real parts.

If a particular partitioning gives rise to an unsymmetric system in equa-
tion (27), i.e., the node-by-node partition (12), the theorem of Bellman
is not applicable. For this case one may invoke Siljak's algebraic criteria
[15] to examine the positive realness of equation (27).

Alternatively, a perturbation theory such as the one expounded by
Vidyasagar [16] can be employed to assess the asymptotic stability of equa-
tion (27) for $\delta \to (0,\infty)$. Stability for intermediate stepsize ranges can
then be determined by a model problem as successfully exploited in [6].

13

If we restrict ourselves to undamped cases, i.e., $\underset{\sim}{B} = \underset{\sim}{0}$ in equation (27), a simple criterion can be derived for the positive realness of equation (27) which involves unsymmetric matrices. To this end we rewrite equation (27) as:

$$\det\left(\begin{bmatrix} \underset{\sim}{A}_x & \underset{\sim}{0} \\ -\delta^2 \underset{\sim}{C}_y^{yx} & \underset{\sim}{A}_y - \delta^2 \underset{\sim}{C}_y^{yy} \end{bmatrix} z^2 + \delta^2 \underset{\sim}{C}\right) = 0 \tag{28}$$

in which

$$\underset{\sim}{C}_y^{yy} = \text{Sym}[\underset{\sim}{C}_y]$$

$$\underset{\sim}{C}_y^{yx} = \underset{\sim}{C}_y - \underset{\sim}{C}_y^{yy} \tag{29}$$

In equation (29), we have $\underset{\sim}{C}_y^{yy} = \underset{\sim}{C}_{yy}$ and $\underset{\sim}{C}_y^{yx} = \underset{\sim}{C}_{yx}$ for partition (12), and $\underset{\sim}{C}_y^{yy} = \underset{\sim}{0}$ for partitions that render I-I procedures. We now state a theorem for equation (28) (see Appendix B).

<u>Theorem 2.</u> If $\underset{\sim}{A}_x$ and $\left(\underset{\sim}{A}_y - \delta^2 \underset{\sim}{C}_y^{yy}\right)$ are non-negative definite and $\underset{\sim}{C}$ is positive definite, then equation (28) has no roots with positive real parts.

<u>Remark:</u> Theorem 2 is applicable to damped cases ($\underset{\sim}{B} \neq \underset{\sim}{0}$) if $\underset{\sim}{B}$ is a Rayleigh-type [17]. This is because $\underset{\sim}{B}$ can be made parts of $\underset{\sim}{A}$ and $\underset{\sim}{C}$ by a commutable matrix transformation [18]. The ensuing stepsize restriction for unsymmetrically partitioned I-E procedures can be obtained by replacing appropriate transformed matrices in Theorem 2.

14

IMPLICIT-EXPLICIT PROCEDURES IN STRUCTURAL DYNAMICS

In the previous section we have presented a technique for determining the stability of the general partitioned procedure (9) through the concept of the solution amplification factor. We now examine the three partitions defined by equations (12) through (14) as applied to structural dynamics problems when the trapezoidal formula and one-term extrapolator (25) are used. It will be shown that the three resulting procedures have algorithmic characteristics identical to those which would result from the combined use of the central-difference formula for the explicit partitions and of the trapezoidal formula for the implicit partitions.

In the sequel we replace the matrices in equation (1) by the more familiar notation:

$$(\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}) = (\underset{\sim}{M}, \underset{\sim}{D}, \underset{\sim}{K}) \tag{30}$$

Node-by-Node I-E Partition

The general characteristic z-polynomial equation (27) for the partition (12) becomes for $\underset{\sim}{D} = 0$:

$$\det [(\underset{\sim}{M} - \delta^2 \underset{\sim}{K}_y) z^2 + \delta^2 \underset{\sim}{K}] = 0 \tag{31}$$

Application of Theorem 2 to equation (31) in conjunction with equations (12) and (29) indicates that this procedure remains stable provided:

$$h \leq 2 / (\omega_{BM})_{max} \tag{32}$$

15

where $\left(\omega_{BM}\right)_{max}$ is the largest eigenvalue of:

$$\omega_{BM}^2 \, \underset{\sim}{M} \, \underline{u} \; = \; \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{K}_{yy} \end{bmatrix} \underline{u} \tag{33}$$

If the trapezoidal and the central difference formulas are used to integrate $(\underline{u}^x, \, \underline{u}^b)^T$ and $\underline{u}^y$, respectively, it can be shown that the resulting character-
istic matrix is:

$$\underset{\sim}{J}_B(\lambda) \; = \; (\lambda - 1)^2 \, \underset{\sim}{M} + \delta^2 \, (\lambda + 1)^2 \, (\underset{\sim}{K} - \underset{\sim}{K}_y) + 4\delta^2 \lambda \underset{\sim}{K}_y \tag{34}$$

It is noted that equation (26), which is obtained by specializing the general
characteristic equation (19) for the trapezoidal formula with one-term extra-
polator, is identical to equation (34).

Element-by-Element I-E Partition

For the partition (13), Bellman's Theorem is directly applicable and provides
the following stability condition:

$$h \leq 2 \, / \, (\omega_{HL})_{max} \tag{35}$$

in which $\left(\omega_{HL}\right)_{max}$ is the largest eigenvalue of:

$$\omega_{HL}^2 \, \underset{\sim}{M} \, \underline{u} \; = \; \begin{bmatrix} 0 & 0 & 0 \\ 0 & \underset{\sim}{K}_{ob}^y & \underset{\sim}{K}_{by} \\ 0 & K_{yb} & \underset{\sim}{K}_{yy} \end{bmatrix} \tag{36}$$

The above stability condition (35) is identical to that obtained by Hughes
and Liu [3] by an energy stability criterion. It can also be shown that
the combined use of the Newmark-family predictor-corrector pairs for the
partition (13) gives the same algorithmic characteristics as equation (34).
The presence of damping can be accounted for to give the same stability con-
dition as that derived in [3].

16

DOF-by-DOF I-E Partition

As the partition (14) renders a symmetric system, we can invoke Bellman's
Theorem to determine its stability. It can be shown that the resulting
stability condition of this partition is identical to that of the node-by-
node partition (32). The algorithmic characteristic equation is obtained
from equation (34) by taking $\underset{\sim}{K}_y$ in the form:

$$
\underset{\sim}{K}_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \underset{\sim}{K}_{yy} \end{bmatrix}
\tag{37}
$$

Remark 1: Examination of the three I-E partitions demonstrates that regard-
less of the particular partition selected, the computer implementation of
the present procedure, equations (3) through (9), can be accomplished within
the framework of implicit integration formulas. Hence, any need for explicit
or predictor formulas to integrate separately the explicit partitions is
eliminated.

Remark 2: An important feature of the general partitioned procedure (9) is
that it allows the combined use of the element-by-element and the DOF-by-DOF
partitions simultaneously. This is possible because the two partitions pre-
serve symmetry of the matrix in the left-hand side in equation (9) and a single
extrapolated vector is applicable to the coupling term in the right-hand
side of equation (9). For example, the element-by-element partition can be
used for partitioning different elements (shells, beams, low- and high-order
isoparametric elements for solid continua) or different media (fluid, struc-
ture, soil subdomains). At the same time, the DOF-by-DOF partition can be
applied within each element or subdomain to integrate the DOF's exhibiting
low (high) frequencies explicitly (implicitly).

17

IMPLICIT-IMPLICIT PROCEDURES

The need for implicit-implicit (I-I) partitioned procedures arises when the coupled-field equations are to be integrated implicitly by sequentially employing single-field integration packages. We will now consider two possible I-I partitioned procedures that may be useful for structure-soil interaction and structure-fluid interaction problems.

Staggered I-I Partition

Let us consider the partitioned matrix:

$$\underset{\sim}{K}_y = \begin{bmatrix} \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{K}_{yb} & \underset{\sim}{0} \end{bmatrix} \tag{38}$$

For this I-I partition the solution-advancing cycle proceeds as follows. First, the vector $\underline{u}^y$ [see equation (11)] is implicitly obtained by extrapolating $\underline{u}^b$. Then use $\underline{u}^y$ to obtain implicitly the vector $(\underline{u}^x, \underline{u}^b)^T$.

Notice that the characteristic z-polynomial equation (28) is applicable to this I-I partition. Applying Theorem 2 to this partition in conjunction with equation (38) indicates that the preceding I-I procedure is unconditionally stable.

This is in striking contrast to the result of Belytschko, Yen and Mullen [8] from their study of a model two-DOF system, in which they proved that the preceding partitioned procedure becomes "unconditionally unstable". We offer the following explanation for this discrepancy.

18

The difference equations employed in [8] are:

$$\underset{\sim}{u}_n^P = \underset{\sim}{u}_{n-1}$$

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x) \, \underset{\sim}{u}_n = 2 \, (\underset{\sim}{M} - \delta^2 \underset{\sim}{K}) \, \underset{\sim}{u}_{n-1} - (\underset{\sim}{M} + \delta^2 \underset{\sim}{K}) \, \underset{\sim}{u}_{n-2} \qquad (39)$$

$$- \delta^2 \underset{\sim}{K}_y \, \underset{\sim}{u}_n^{(P)}$$

The above difference equation (39) can be written equivalently by a pair of difference equations and the integrator for velocity as:

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x) \, \underset{\sim}{u}_n = (\underset{\sim}{M} - \delta^2 \underset{\sim}{K}) \, \underset{\sim}{u}_{n-1} + 2 \, \delta \, \underset{\sim}{M} \, \dot{\underset{\sim}{u}}_{n-1}$$

$$(40)$$

$$\underset{\sim}{M} \, \dot{\underset{\sim}{u}}_n = \underset{\sim}{M} \, \dot{\underset{\sim}{u}}_{n-1} - \delta \, \underset{\sim}{K} \, (\underset{\sim}{u}_{n-1} + \underset{\sim}{u}_n)$$

On the other hand, our procedure (9), when specialized to the case of the trapezoidal formula and one-term extrapolation (35), becomes:

$$\underset{\sim}{u}_n^{(P)} = \underset{\sim}{u}_{n-1}$$

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x) \, \underset{\sim}{u}_n = (\underset{\sim}{M} - \delta^2 \underset{\sim}{K}) \, \underset{\sim}{u}_{n-1} + 2 \, \delta \, \underset{\sim}{M} \, \dot{\underset{\sim}{u}}_{n-1} - \delta^2 \underset{\sim}{K}_y \, \underset{\sim}{u}_n^{(P)} \quad (41)$$

$$\underset{\sim}{M} \, \dot{\underset{\sim}{u}}_n = \underset{\sim}{M} \, \dot{\underset{\sim}{u}}_{n-1} - \delta \, \underset{\sim}{K} \, (\underset{\sim}{u}_{n-1} + \underset{\sim}{u}_n)$$

Comparing equations (40) with (41) one sees that the latter is equivalent to the former if $\underset{\sim}{u}^{(P)} = 0$ in equation (41). The characteristic z-polynomial equation for equation (40) can be derived as

$$\det \left[\left(\underset{\sim}{M} - \tfrac{1}{2} \, \delta^2 \underset{\sim}{K}_y \right) z^2 - \frac{\delta^2}{2} \underset{\sim}{K}_y \, z + \delta^2 \underset{\sim}{K} \right] = 0 \qquad (42)$$

from which one can establish that the procedure is "unconditionally unstable" for the model problem considered in [8].

<u>Remark</u>: If the extrapolator in equation (39) is changed to

$$u_{-n}^{(p)} = 2u_{-n-1} - u_{-n-2} \qquad (43)$$

it can be shown that the resulting characteristic equation is identical
to that of the present procedure (41). This is a direct consequence of
the well-known property that the order of accuracy of difference equa-
tions for second-order systems maintains that of the integration formula
when discretized by one-derivative formulas and it decreases by one from
that of the integration formula when discretized by two-derivative form-
ulas. Therefore, from the viewpoint of our procedure (41) the procedure
of [8] corresponds to neglecting the coupling term when computing $\underline{u}^y$.

Although the present staggered I-I procedure (41) which is a special
case of the general procedure (9) is unconditionally stable, it
does not meet the modularity requirement. The reason for this is that the
entries in the matrix $\underline{K}_y$ in equation (38) cannot be decomposed into the
two subdomains as shown in Figure 1. The remedy to this shortcoming is an
element-by-element I-I partitioned procedure.

Element-by-Element I-I Partition

If the partitioned matrix $\underline{E}_y$ in equation (9) contains non-zero diagonal
entries, the solution vector pertaining to $\underline{E}_y$ is necessarily obtained by
an explicit process as shown in the section on Implicit-Explicit Proced-
ures. One can overcome this deficiency by implicitly solving two sets of
difference equations for both partitioned subdomains (see Figure 1).
This is accomplished by the following I-I procedure:

$$(\underline{M} + \delta^2 \underline{K}_x) \, \underline{u}_{-nt} = \underline{g}_n - \delta^2 \underline{K}_y \, \underline{u}_{-n-1}$$

$$(\underline{M} + \delta^2 \underline{K}_y) \, \underline{u}_n = \underline{g}_n - \delta^2 \underline{K}_x \, \underline{u}_{-nt} \qquad (44)$$

where $\underline{K}_y$ takes the partition (13).

20

For the particular case: $\underset{\sim}{A} = \underset{\sim}{M}$, $\underset{\sim}{E}_x = \underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x$, and $\underset{\sim}{E}_y = \delta^2 \underset{\sim}{K}_y$, the stability of the element-by-element I-I procedure (44) can be analyzed and the resulting z-polynomial stability equation is (see Appendix C):

$$
\det \left[\left(4\underset{\sim}{M} + 2 \delta^4 \left[\underset{\sim}{K}_x \underset{\sim}{M}^{-1} \underset{\sim}{K}_y + \underset{\sim}{K}_y \underset{\sim}{M}^{-1} \underset{\sim}{K}_x \right] \right) z^2 \right.
$$
$$
\left. + 2 \delta^4 \left(\underset{\sim}{K}_x \underset{\sim}{M}^{-1} \underset{\sim}{K}_y - \underset{\sim}{K}_y \underset{\sim}{M}^{-1} \underset{\sim}{K}_x \right) z + 4 \delta^2 \underset{\sim}{K} \right] = 0
$$

$$(45)$$

The first and the third coefficient matrices are positive definite. The second matrix $(\underset{\sim}{K}_x \underset{\sim}{M}^{-1} \underset{\sim}{K}_y - \underset{\sim}{K}_y \underset{\sim}{M}^{-1} \underset{\sim}{K}_x)$ is antisymmetric and therefore has purely imaginary eigenvalues only. Hence, the element-by-element I-I procedure (44) is stable by Theorem 1.

It is noted that the preceding I-I partitioned procedure (44) preserves the program modularity of each single-field implicit integration package. As the preceding I-I procedure is new we will describe its computational aspects in some detail.

First, equation (44a) is implicitly solved for the vector $(u_{-nt}^x, u_{-nt}^b)^T$ by utilizing the extrapolated vector $(u_{-n-1}^b, u_{-n-1}^y)^T$. Then solve equation (44b) for the vector $(u_{-n}^b, u_{-n}^y)^T$ by utilizing the temporary solution vector $(u_{-nt}^x, u_{-nt}^b)^T$. The final solution vector to be used for the next step integration is thus $(u_{-nt}^x, u_{-n}^b, u_{-n}^y)^T$. In the next step equation (44b) is solved first. The integration process is continued by solving equations (44a) and (44b) alternatively.

From the foregoing examination of the computational sequence of the element-by-element I-I procedure (44), it is evident that two implicit solutions are required for the boundary vector u^b per step. Therefore, the price paid for realizing the computational modularity by the element-by-element I-I procedure is one additional implicit solution of the boundary vector $\underset{-}{u}^b$ as compared to that of the staggered I-I procedure (41).

21

DISCUSSION AND CONCLUSIONS

In this paper the complete set of coupled-field equations of motion is discretized by implicit integration formulas. A general partition is then introduced to the resulting implicit difference equations. Finally, the coupling terms that end up in the right-hand side as unknowns are extrapolated to advance the solution cycle. The ensuing stability analysis for the general partitioned procedure (9) is carried out via the concept of solution amplification of the difference equations. We now summarize the main results of the present study.

It is demonstrated that a unified stability analysis is applicable to all existing and some new partitioned procedures even if the matrix equations that describe the stability characteristics of the partitioned procedures are not simultaneously diagonalizable. This applicability is in fact due to the equivalence between the well-known energy technique [19] and the positive definiteness of quadratic polynomials in the matrix theory that underlies the theorems of Routh-Hurwitz and Lyapunov.

Our partitioned procedure (9) can be specialized to a general implicit-explicit procedure that embodies those of Belythshko and Mullen [1-2] and Hughes and Liu [3-4]. In particular, it led to the discovery of a new DOF-by-DOF implicit-explicit procedure that preserves the symmetry of the partitioned matrices and allows degree-by-degree implicit or explicit selections of the solution vector. Furthermore, our implicit-explicit procedure can accommodate, within the framework of implicit integration implementation, the concurrent use of the element-by-element and the DOF-by-DOF I-E procedures.

22

Our procedure (9) becomes the staggered I-I procedure (41) when the partitioned matrix does not contain non-zero diagonal entries. The staggered I-I procedure is unconditionally stable if the extrapolator is judiciously selected. On the other hand, the cause of unconditional instability of the I-I procedure of Belytschko, Yen and Mullen [8] can be traced to an improper choice of the extrapolation vector.

A new element-by-element I-I procedure (44) is presented, which preserves the program modularity of each single-field integration package when used to solve coupled-field problems. The procedure is stable and its only computational overhead as compared to that of the staggered I-I procedure is the requirement of an additional implicit solution of the boundary state vector.

So far we have proposed and analyzed procedures that integrate field solution vectors with the same time step size. The stability analysis of procedures which integrate field solution vectors with different time step sizes is much more involved and the resulting computational gain margins are in general harder to assess. Study of such fractional-step partitioned procedures is now being actively pursued and will be reported in a separate communication.

ACKNOWLEDGMENTS

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25

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APPENDIX A

DERIVATION OF CHARACTERISTIC POLYNOMIAL EQUATION

The linear multistep formula (2) can be expressed in the form

$$\sum_{j=0}^{m} (\alpha_j \, \underline{u}_{n-j} - \delta \, \beta_j \, \underline{\dot{u}}_{n-j}) = 0 \qquad\qquad (A.1)$$

and a stable form of extrapolators is found to be [6]:

$$\underline{u}_n^{(P)} = \sum_{j=1}^{m} \hat{\alpha}_j \, \underline{u}_{n-j} \qquad\qquad (A.2)$$

The historical vector $\underline{h}_n^u$ in equation (2) is

$$\underline{h}_n^u = \sum_{j=1}^{m} (\delta \, \beta_j \, \underline{\dot{u}}_{n-j} - \alpha_j \, \underline{u}_{n-j}) \qquad\qquad (A.3)$$

provided $\alpha_o = 1$, which we assume.

Equations (A.1-3) can be expressed in terms of the characteristic value λ of the difference Equation (10) and $\underline{u}_{n-m}$, viz.:

$$\rho(\lambda) \, \underline{u}_{n-m} - \delta \, \sigma(\lambda) \, \underline{\dot{u}}_{n-m} = 0$$

$$\underline{u}_n^{(P)} = e(\lambda) \, \underline{u}_{n-m} \qquad\qquad (A.4)$$

$$\underline{h}_n^u = \delta[\sigma(\lambda) - \lambda^m] \, \underline{\dot{u}}_{n-m} + [\lambda^m - \rho(\lambda)] \, \underline{u}_{n-m}$$

A-1

The velocity term (A.8) can only be eliminated by differentiation formula resulting from (A.4):

$$\dot{\underline{u}}_{n-m} = (\rho/\delta\sigma)\,\underline{u}_{n-m} \tag{A.9}$$

which is substituted into (A.8) to yield:

$$[\rho^2\,\underline{A} + \delta\sigma\rho\,\underline{B}_x + \delta^2\,\sigma^2\,\underline{C}_x + \delta(\rho\sigma - \lambda^m\rho + \rho e)\,\underline{B}_y$$
$$+ \delta^2(\sigma^2 - \lambda^m\rho + \rho e)\,\underline{C}_y]\,\underline{u}_{n-m} = 0 \tag{A.10}$$

It should be noted that if the differentiation formula (A.9) is used in place of (A.6), the following equation results:

$$[\rho^2\,\underline{A} + \delta\,\rho\sigma\,\underline{B}_x + \delta(\rho\sigma - \lambda^m\sigma + \sigma e)\,\underline{B}_y$$
$$+ \delta^2\,\sigma^2\,\underline{C}_x + \delta^2(\sigma^2 - \lambda^m\sigma + \sigma e)\,\underline{C}_y]\,\underline{u}_{n-m} = 0 \tag{A.11}$$

This has roots whose magnitudes are greater than unity in general for consistent integration formulas; thus the integration process is unstable. The reader can verify this assertion for the trapezoidal rule and one-term extrapolator.

APPENDIX B

PROOF OF THEOREM 2

Let us consider the undamped system from equation (1)

$$\underset{\sim}{A} \, \ddot{\underline{u}} + \underset{\sim}{C} \, \underline{u} \; = \; \underline{0} \tag{B.1}$$

Equation (B.1) can be transformed to an equivalent form

$$\ddot{\underline{q}} + \hat{\underset{\sim}{C}} \, \underline{q} \; = \; \underline{0}$$

where

$$\underline{q} \; = \; \underset{\sim}{L}^{-T} \, \underline{u}$$

$$\underset{\sim}{A} \; = \; \underset{\sim}{L} \, \underset{\sim}{L}^{T} \text{ and } \hat{\underset{\sim}{C}} \; = \; \underset{\sim}{L}^{-1} \, \underset{\sim}{C} \, \underset{\sim}{L}^{-T} \tag{B.2}$$

The characteristic system for (B.2) which corresponds to equation (28) can be derived as

$$z^2 \, \underset{\sim}{G} \, \underline{q} + \delta^2 \, \hat{\underset{\sim}{C}} \, \underline{q} \; = \; \underline{0}$$

where

$$\underset{\sim}{G} \; = \; \begin{bmatrix} \underset{\sim}{I} & \underline{0} \\ -\delta^2 \hat{\underset{\sim}{C}}_{\underset{\sim}{y}}^{yx} & \underset{\sim}{I} - \delta^2 \hat{\underset{\sim}{C}}_{\underset{\sim}{y}}^{yy} \end{bmatrix} \tag{B.3}$$

Equation (B.3) can be further reduced by the orthogonal transformation

$$\underline{q} \; = \; \begin{bmatrix} \underset{\sim}{I} & 0 \\ 0 & \underset{\sim}{T} \end{bmatrix} \underline{v} \tag{B.4}$$

B-1

so that

$$\underset{\sim}{T}^T \hat{\underset{\sim}{C}}_{yy} \underset{\sim}{T} = \underset{\sim}{\Lambda} = \text{diag}\left(\Omega_1^2, \ldots, \Omega_k^2\right) \qquad (B.5)$$

where k is the size of $\underset{\sim}{C}_{yy}$.

Using equations (B.4) and (B.5), one transforms equation (B.3) to

$$z^2 \hat{\underset{\sim}{G}} \underset{\sim}{v} + \delta^2 \hat{\hat{\underset{\sim}{C}}} \underset{\sim}{v} = 0$$

where

$$\hat{\underset{\sim}{G}} = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ -\delta^2 \underset{\sim}{T}^T \hat{\underset{\sim}{C}}_y^{yx} & \underset{\sim}{I} - \delta^2 \underset{\sim}{T}^T \hat{\underset{\sim}{C}}_y^{yy} \underset{\sim}{T} \end{bmatrix}$$

$$\hat{\hat{\underset{\sim}{C}}} = \begin{bmatrix} \hat{\underset{\sim}{C}}_{xx} & \underset{\sim}{B} \\ \underset{\sim}{B}^T & \underset{\sim}{\Lambda} \end{bmatrix} \qquad (B.6)$$

in which $\underset{\sim}{B} = \hat{\underset{\sim}{C}}_{xy} \underset{\sim}{T}$.

Case: $\hat{\underset{\sim}{C}}_y^{yy} = \hat{\underset{\sim}{C}}_{yy}$ and $\hat{\underset{\sim}{C}}_y^{yx} = \hat{\underset{\sim}{C}}_{yx}$. For this case we have

$$\underset{\sim}{T}^T \hat{\underset{\sim}{C}}_y^{yx} = \underset{\sim}{B}^T$$

$$\underset{\sim}{T}^T \hat{\underset{\sim}{C}}_y^{yy} \underset{\sim}{T} = \underset{\sim}{\Lambda} \qquad (B.7)$$

Now, equation (B.6a) can be expressed in the form

$$z^2 \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} - \delta^2 \underset{\sim}{\Lambda} \end{bmatrix} \hat{\underset{\sim}{v}} + \delta^2 \underset{\sim}{H}^{-1} \hat{\hat{\underset{\sim}{C}}} \underset{\sim}{H} \hat{\underset{\sim}{v}} = 0 \qquad (B.8)$$

where $\underset{\sim}{H}$ is a similarity transformation matrix

$$\underset{\sim}{H} = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ -\underset{\sim}{\Lambda}^{-1} \underset{\sim}{B}^T & \underset{\sim}{I} \end{bmatrix} \qquad (B.9)$$

Note that if $(\underset{\sim}{I} - \underset{\sim}{\delta}^2 \underset{\sim}{\Lambda})$ and $\hat{\underset{\sim}{C}}$ are positive definite, all z_i^2, $i = 1,\ldots, N$ are negative real as the similarity transformation preserves the eigenvalues of original system (B.6a).

Case: $\hat{\underset{\sim}{C}}_y^{yy} = \underset{\sim}{0}$, $\hat{\underset{\sim}{C}}_y^{yx} = \underset{\sim}{C}_{yx}$. In this case, we have

$$z^2 \, \hat{\underset{\sim}{G}}^I \, \underline{v} + \delta^2 \, \hat{\underset{\sim}{C}} \, \underline{v} = 0$$

where

$$\hat{\underset{\sim}{G}}^I = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ -\delta^2 \underset{\sim}{B}^T & \underset{\sim}{I} \end{bmatrix} \tag{B.10}$$

As $\hat{\underset{\sim}{G}}^I$ is not diagonalizable, we perturb it to the form

$$\hat{\underset{\sim}{G}}^{IP} = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ -\delta^2 \underset{\sim}{B}^T & \underset{\sim}{I} + \delta^2 \varepsilon \underset{\sim}{I} \end{bmatrix} \tag{B.11}$$

where $0 < \delta^2 \varepsilon \ll 1$. Substituting the perturbed matrix (B.11) into (B.10a) and following similar steps as used in deriving equation (B.6a), one can obtain

$$z^2 \begin{bmatrix} I & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{I} + \delta^2 \varepsilon \underset{\sim}{I} \end{bmatrix} \hat{\underline{v}} + \underset{\sim}{H}^{-P} \, \hat{\underset{\sim}{C}} \, \underset{\sim}{H}^P \, \hat{\underline{v}} = 0 \tag{B.12}$$

where

$$\underset{\sim}{H}^P = \begin{bmatrix} \underset{\sim}{I} & \underset{\sim}{0} \\ \frac{1}{\varepsilon} \underset{\sim}{B}^T & \underset{\sim}{I} \end{bmatrix} \tag{B.13}$$

We note that the eigenvalues of a matrix depend continuously on the elements of the matrix (see, e.g., Vidyasagar [16]). Therefore, as $\varepsilon \to 0$, equation (B.12) contains the eigenvalues of the system (B.10). Furthermore, $\underset{\sim}{H}^P$ is a similarity transformation matrix; hence, if $\hat{\underset{\sim}{C}}$ is positive definite, all eigenvalues z_i^2, $i = 1,\ldots, N$ in equation (B.10) are negative real. This completes the proof of Theorem 2.

APPENDIX C

DERIVATION OF THE STABILITY Z-POLYNOMIAL
EQUATION FOR ELEMENT I-I PARTITION (44)

The difference equation (44) for the trapezoidal formula with one-term extrapolation becomes:

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x)\, \underline{u}_n^{(0)} = (\underset{\sim}{M} - \delta^2 \underset{\sim}{K})\, \underline{u}_{n-1} + 2\delta \underset{\sim}{M}\, \underline{\dot{u}}_{n-1} - \delta^2 \underset{\sim}{K}_y\, \underline{u}_{n-1}$$

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_y)\, \underline{u}_n = (\underset{\sim}{M} - \delta^2 \underset{\sim}{K})\, \underline{u}_{n-1} + 2\delta \underset{\sim}{M}\, \underline{\dot{u}}_{n-1} - \delta^2 \underset{\sim}{K}_x\, u_n^{(0)}$$

(C.1)

from which one obtains

$$\underset{\sim}{M}\, \underline{u}_n^{(0)} = (\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_y)\, \underline{u}_n - \delta^2 K_y\, \underline{u}_{n-1}$$

(C.2)

for the time step $t = t_{n+1}$, we alternate the integration as

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_y)\, \underline{u}_{n+1}^{(0)} = (\underset{\sim}{M} - \delta^2 \underset{\sim}{K})\, \underline{u}_n + 2\delta\, \underline{\dot{u}}_n - \delta^2 \underset{\sim}{K}_x\, \underline{u}_n$$

$$(\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x)\, \underline{u}_{n+1} = (\underset{\sim}{M} = \delta^2 \underset{\sim}{K})\, \underline{u}_n + 2\delta\, \underline{\dot{u}}_n - \delta^2 \underset{\sim}{K}_y\, \underline{u}_{n+1}^{(0)}$$

(C.3)

which gives

$$\underset{\sim}{M}\, \underline{u}_{n+1}^{(0)} = (\underset{\sim}{M} + \delta^2 \underset{\sim}{K}_x)\, \underline{u}_{n+1} - \delta^2 \underset{\sim}{K}_x\, \underline{u}_n$$

(C.4)

Now, substitute equations (C.2) and (C.4) into equations (C.1b) and (C.3b), respectively, and subtract the two resulting equations to yield:

$$\underset{\sim}{M}(\underline{u}_{n+1} - \underline{u}_n) + \delta^2 \underset{\sim}{K}_x\, \underline{u}_{n+1} - \delta^2 \underset{\sim}{K}_y\, \underline{u}_n = (\underset{\sim}{M} - \delta^2 \underset{\sim}{K})(\underline{u}_n - \underline{u}_{n-1})$$

$$+ 2\delta\, \underset{\sim}{M}(\underline{\dot{u}}_n - \underline{\dot{u}}_{n-1}) + \delta^4 \underset{\sim}{K}_y\, \underset{\sim}{M}^{-1} \underset{\sim}{K}_x\, \underline{u}_n - \delta^4 \underset{\sim}{K}_y\, \underset{\sim}{M}^{-1} \underset{\sim}{K}_y\, \underline{u}_{n-1}$$

(C.5)

The velocity terms in equation (C.5) can be expressed by the integration formula and the equations of motion as:

$$\underset{\sim}{M}\left(\dot{\underline{u}}_n - \dot{\underline{u}}_{n-1}\right) \;\; = \;\; - \delta\underset{\sim}{K}\left(\underline{u}_n + u_{n-1}\right) \tag{C.6}$$

Eliminating the term $(\dot{\underline{u}}_n - \dot{u}_{n-1})$ in equation (C.5) by equation (C.6) one obtains

$$\underset{\sim}{M}\left(\underline{u}_{n+1} - 2\underline{u}_n + \underline{u}_{n-1}\right) + \delta^2\underset{\sim}{K}\left(\underline{u}_n + 2\underline{u}_{n-1} + \underline{u}_n\right)$$

$$+ \delta^4\left[\underset{\sim}{K}_y\underset{\sim}{M}^{-1}\underset{\sim}{K}_x\, \underline{u}_{n+1} - \left(\underset{\sim}{K}_x\underset{\sim}{M}^{-1}\underset{\sim}{K}_y + \underset{\sim}{K}_y\underset{\sim}{M}^{-1}\underset{\sim}{K}_x\right)\underline{u}_n + \underset{\sim}{K}_x\underset{\sim}{M}^{-1}\underset{\sim}{K}_y\, \underline{u}_{n-1}\right] = 0 \tag{C.7}$$

Finally, transformation of λ into z via equation (22) gives the desired result, viz:

$$\det\left[\left(4\underset{\sim}{M} + 2\underset{\sim}{K}_y\underset{\sim}{M}^{-1}\underset{\sim}{K}_x + 2\underset{\sim}{K}_x\underset{\sim}{M}^{-1}\underset{\sim}{K}_y\right)z^2 + 2\delta^4\left(\underset{\sim}{K}_x\underset{\sim}{M}^{-1}\underset{\sim}{K}_y - \underset{\sim}{K}_y\underset{\sim}{M}^{-1}\underset{\sim}{K}_x\right)z\right.$$

$$\left. + 4\delta^2\underset{\sim}{K}\right] \;=\; 0 \tag{C.8}$$

It is noted that the integration sequence which starts with the extrapolated term $\underset{\sim}{K}_x\,\underline{u}_{n-1}$ in equation (C.1) changes the sign of the second matrix in equation (C.8). However, as the eigenvalues of $\underset{\sim}{K}_x\underset{\sim}{M}^{-1}\underset{\sim}{K}_y$ are the same as those of $\underset{\sim}{K}_y\underset{\sim}{M}^{-1}\underset{\sim}{K}_x$ the following equality holds:

$$\underline{x}^T\underset{\sim}{K}_y\underset{\sim}{M}^{-1}\underset{\sim}{K}_x\underline{x} - \underline{x}^T\underset{\sim}{K}_x\underset{\sim}{M}^{-1}\underset{\sim}{K}_y\underline{x} \;=\; 0 \tag{C.9}$$

for an arbitrary vector $\underline{x}$. Therefore, the I-I partition (44) is stable.